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A GENERAL VARIATIONAL PRINCIPLE FOR THERMAL INSULATION SYSTEM DESIGN

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Abstract—The paper proposes a conceptually new method for thermal insulation system optimization. The method, based on minimizing thermodynamic irreversibility, consists of externally controlling the variation of heat leak with temperature across the insulation. It is demonstrated that the useful power savings registered from applying this design philosophy are important. Prime candidates for this method are insulation systems facing high absolute temperature ratios and insulation systems in which the effective thermal conductivity increases with the absolute temperature. Three classes of thermal insulation systems are optimized based on this universal design procedure: one-dimensional continuous insulations, insulations with discontinuous temperature distribution and continuous insulations with internal heat generation.

NOMENCLATURE

- A, insulation area;
- b, electrical resistivity constant;
- c_p, specific heat at constant pressure;
- F, factor, equation (26);
- i, radiation shield position;
- I, electrical current;
- k, effective thermal conductivity;
- k_l, constant effective thermal conductivity;
- $\dot{m}$, mass flowrate;
- n, exponent, equation (19);
- N, number of gaps between radiation shields;
- P, wetted perimeter;
- $q(T)$, heat leak function;
- $r(T)$, intermediate heat exchange function;
- S_{gen}, rate of entropy generation;
- t, insulation thickness (heat leak path length);
- T, absolute temperature;
- T_0, environment temperature;
- T_1, low temperature;
- T_2, high temperature;
- $T^{(i)}$, radiation shield temperature;
- ΔT, temperature difference, equation (21);
- U, overall heat-transfer coefficient, equation (23);
- x, transversal coordinate.

Greek symbols

- λ, Λ, Lagrange multipliers;
- ρ, electrical resistivity;
- σ, Boltzmann's constant;
- τ, ratio of absolute temperatures, T_2/T_1;
- $\tau^{(i)}$, optimum shield temperature, $T^{(i)}_{opt}/T_1$;
- Φ, integral, equations (6) and (36).

Superscript

- $*$, insulation design with q = constant.

Subscripts

- min, minimum;
- opt, optimum.

1. INTRODUCTION

THERMAL insulation is perhaps the simplest system encountered in thermal design. From the point of view of most designers, the basic function of thermal insulation is to limit the flow of heat between two neighboring surfaces at different temperatures. This is a first law point of view which is accepted widely, as demonstrated by almost anything written or taught on the subject.

An alternative way of looking at thermal insulation systems is presented in Fig. 1(a). Thermal insulations are in essence dissipators of useful mechanical power for one can always envision putting the insulation temperature difference to work. It will be demonstrated that as the ratio of absolute temperatures bordering the insulation increases, one cannot afford to neglect the dissipative effect distributed throughout the insulation. It will also be demonstrated that the use of second law concepts is in fact a surprisingly simple and powerful analytical approach to decisions in thermal design.

The objective of this work is to propose a general optimization philosophy for thermal insulation systems. The method relies on the second law of thermodynamics and its related concept, irreversibility or entropy generation. This work will show that there are two ways in which the performance of an insulation system can be improved. The first is the traditional approach which consists of decreasing the thermal conductance or the heat leak penetrating the insulation. The second approach is novel and applies to an insulation system whose thermal conductance is fixed, i.e. it has already been reduced to the lowest value acceptable economically. This approach con-

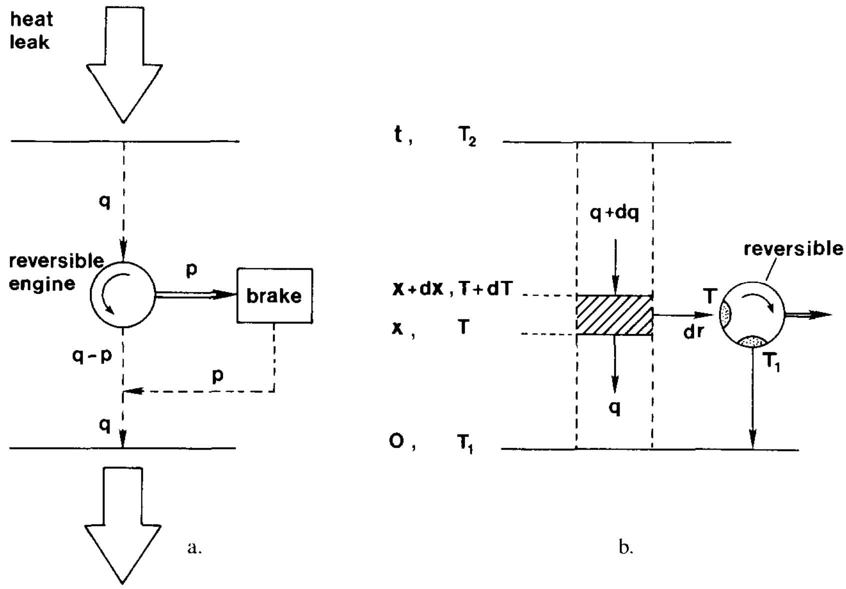

FIG. 1. Schematic of one-dimensional thermal insulation system showing internal dissipation of mechanical power and entropy generation analysis.

sists of controlling the temperature variation of heat leak across the insulation, by cooling or even heating the insulation at intermediate temperatures. The objective of this paper is to demonstrate the second approach.

As a summary to what follows, in Section 2 we consider the optimum design of one-dimensional continuous insulations. The bulk of the paper is devoted to demonstrating the general applicability of the design philosophy proposed in Section 1. In Sections 3 and 4 we apply the same method to insulations with discontinuous temperature distribution and insulations with internal heat generation, respectively. The potential savings in useful power derived from implementing this design philosophy are critically assessed.

2. ONE-DIMENSIONAL CONTINUOUS INSULATIONS

Insulation irreversibility (entropy generation)

Consider the one-dimensional thermal insulation system of Fig. 1(b). A layer of thickness t and cross-sectional area A separates two parallel surfaces with different absolute temperatures, T_1, T_2. The temperature varies continuously across the insulation. In any plane x, the heat current q is proportional to the local temperature gradient and an effective thermal conductivity factor k,

$$q = Ak\frac{\mathrm{d}T}{\mathrm{d}x}, \tag{1}$$

where k may be temperature-dependent.

The temperature T of any length segment $\mathrm{d}x$ is controlled externally by placing the insulation segment in local thermal equilibrium with the T end of a reversible (Carnot) device operating between T and T_1. The $\mathrm{d}x$ segment and the reversible device exchange heat at a rate $\mathrm{d}r$. This heat-transfer process is reversible since the $\mathrm{d}x$ segment and the device adjacent to it are at the same temperature. We show later that in practical energy systems the lateral heat-transfer effect $\mathrm{d}r$ may be accomplished by placing the insulation in local thermal equilibrium with the warm end or the cold end of power or refrigeration cycles built into the system to be insulated, as sketched on Fig. 3.

Under these circumstances, the heat flow q through the shaded element of Fig. 1(b) is the only source of thermodynamic irreversibility since it occurs across a finite temperature difference $\mathrm{d}T$. The local rate of entropy generation in the element is [1].

$$\mathrm{d}S_{\text{gen}} = q\frac{\mathrm{d}T}{T^2} = -q\,\mathrm{d}\left(\frac{1}{T}\right). \tag{2}$$

Expression (2) is the result of an entropy flux analysis of the shaded element, taking into account that

$$\mathrm{d}q = \mathrm{d}r, \tag{3}$$

based on the first law of thermodynamics. The total rate of entropy generation in the one-dimensional insulation of Fig. 1(b) is obtained by integrating expression (2) over the temperature interval occupied by the insulation,

$$S_{\text{gen}} = \int_{T_1}^{T_2} q\frac{\mathrm{d}T}{T^2}. \tag{4}$$

Integral (4) is a quantitative measure of the insulation irreversibility, stating how imperfect the insulation is as a thermodynamic system. If T_0 is the environment temperature, then the product T_0S_{gen} represents the share of useful mechanical power dissipated (wasted) due to insulation irreversibility [2, 3]. It is important to minimize the rate of entropy generation, particularly in insulation systems associated with large scale power plants and refrigeration systems where the power savings derived from a reduced S_{gen} are significant.

Minimization of irreversibility in insulations with fixed geometry

We will now consider techniques for reducing the insulation irreversibility. There are a few ways in which this can be accomplished, the immediately obvious being suggested by equation (1). Increasing the layer thickness (t) and/or decreasing the effective conductivity of the insulating material (k) leads to a smaller heat current (q) and, based on equation (4), a better insulation (lower rate of entropy generation). This method is effective in its own right and motivates the continuing quest for inexpensive low thermal conductivity materials and low heat thermal insulation systems, particularly for low temperature applications [4].

A more subtle way of minimizing S_{gen} applies to insulations of fixed geometry and constitution, i.e. systems in which the thermal resistance to transverse heat transfer has already been increased to the limit imposed by economic considerations. Consider, for example, the system of Fig. 1b with k, t and A fixed. Analytically, the fixed geometry condition can be expressed as an integral constraint obtained by rearranging and integrating equation (1).

$$\frac{t}{A} = \int_{T_1}^{T_2} \frac{k(T)}{q(T)} \mathrm{d}T. \tag{5}$$

Constraint (5) takes into account the fact that, due to the heat transfer interaction dr with the Carnot device, the local heat leak q may vary with the absolute temperature. It is the local heat transfer dr which provides the additional degree of freedom for being able to change $q(T)$ and S_{gen} in an insulation of fixed identity.

We are interested in deriving the optimum heat current distribution $q_{opt}(T)$ which minimizes the S_{gen} integral (4) subject to the integral constraint (5). This is an interesting variational problem which consists of finding the function $q(T)$ which minimizes the integral

$$\Phi = \int_{T_1}^{T_2} \left(\frac{q}{T^2} + \lambda \frac{k}{q} \right) \mathrm{d}T, \tag{6}$$

where λ is a Lagrange multiplier. Employing the calculus of variations, the integral of equation (6) is minimized if the unknown function $q(T)$ satisfies the Euler equation for an extremal [5]

$$\frac{1}{q^2} - \frac{\lambda k}{q^2} = 0. \tag{7}$$

The optimum distribution of heat current, q_{opt}, found by solving equation (7) depends on λ. The Lagrange multiplier is finally identified by substituting $q_{opt}(T, \lambda)$ into constraint (5) and solving the resulting equation in λ. Thus, we obtain

$$q_{opt}(T) = T(k)^{1/2} \left[\frac{A}{t} \int_{T_1}^{T_2} \frac{(k)^{1/2}}{T} \mathrm{d}T \right]. \tag{8}$$

The variational result (8) proves that in order to further optimize a thermal insulation system of fixed geometry one has to monitor the heat leak variation with temperature. A system or the traditional type with constant q across the insulating layer is not necessarily the best. Depending on the $k(T)$ function, it may be necessary to either cool (d$r>0$) or, in exceptional cases, even heat (d$r<0$) the insulation at intermediate temperatures in order to approach the optimum prescribed by equation (8).

Combining $q_{opt}(T)$ with equation (1) one can determine the optimum temperature distribution across the insulation, $T_{opt}(x)$. Similarly, the optimum distribution of heat removal effect $r_{opt}(T)$ can be derived by combining these results with equation (3). It is also possible to estimate the minimum reached by S_{gen} when $q \equiv q_{opt}$:

$$S_{gen}^{min} = \frac{A}{t} \left[\int_{T_1}^{T_2} \frac{(k)^{1/2}}{T} \mathrm{d}T \right]^2. \tag{9}$$

Integral (9) can be carried out explicitly as soon as the insulation system is specified and function $k(T)$ becomes known.

Insulations with controlled distribution of heat leak

Here we point out the practical implications of the design principle developed above. Consider the case where the thermal conductivity is constant $k = k_1$. Applying the optimum (8), the best insulation system design is described by

$$q_{opt} = \left(\frac{Ak_1}{t} \ln \frac{T_2}{T_1} \right) T, \tag{10}$$

$$S_{gen}^{min} = \frac{Ak_1}{t} \left(\ln \frac{T_2}{T_1} \right)^2 \tag{11}$$

$$T_{opt} = T_1 \exp \left(\frac{x}{t} \ln \frac{T_2}{T_1} \right), \tag{12}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}T} \right)_{opt} = \frac{Ak_1}{t} \ln \frac{T_2}{T_1}. \tag{13}$$

We find that the optimum design is characterized by a heat leak q which decreases linearly with the decreasing local absolute temperature. The optimum distribution $q_{opt}(T)$ is achieved by removing a constant share of q at each temperature interval as shown in equation (13).

It is interesting to compare the best design (10)–(13) with the case in which no lateral cooling dr is being provided as in all systems traditionally designed for the lowest achievable heat leak. The traditional design, indicated here with an asterisk, is

$$q^* = \frac{Ak_1}{t}(T_2 - T_1), \tag{14}$$

$$S_{gen}^* = \frac{Ak_1}{t} \frac{(T_2 - T_1)^2}{T_1 T_2} \tag{15}$$

$$T^* = T_1 + \frac{x}{t}(T_2 - T_1), \tag{16}$$

$$r^* = 0. \tag{17}$$

Dividing equations (15) and (11) side by side we obtain a relative measure of how inferior design

(14)–(17) is with respect to the optimum (10)–(13),

$$\frac{S^*_{gen}}{S^{min}_{gen}} = \frac{1}{\tau}\left(\frac{\tau-1}{\ln\tau}\right)^2. \quad (18)$$

In equation (18), τ is the absolute temperature ratio T_2/T_1. Expression (18) is shown as the $n = 0$ curve on Fig. 2. We see that only in the limit $\tau \to 1$ the constant q system performs as well as the optimum. Its performance deteriorates considerably as the temperature ratio τ exceeds 10.

$r(T)$ is the only approach to a better thermal performance only in the limit $\tau \to 1$. As soon as the temperature difference $(T_2 - T_1)$ is of the same order of magnitude or greater than T_1, the thermodynamic performance of a given insulation (fixed Ak_1/t) can be improved substantially by appropriately controlling the local heat leak dependence on absolute temperature, $q(T)$.

Figure 2 is a comparison of only one practical design (q = constant) with the theoretical optimum developed in the preceding paragraphs. As illustrated by the variational approach to determining the optimum design, there exists an infinity of designs in which $q(T)$ may or may not come close to $q_{opt}(T)$. The ability of achieving a heat leak distribution $q(T)$ which closely resembles $q_{opt}(T)$ depends on the amount and distribution of lateral heat-transfer effect $r(T)$.

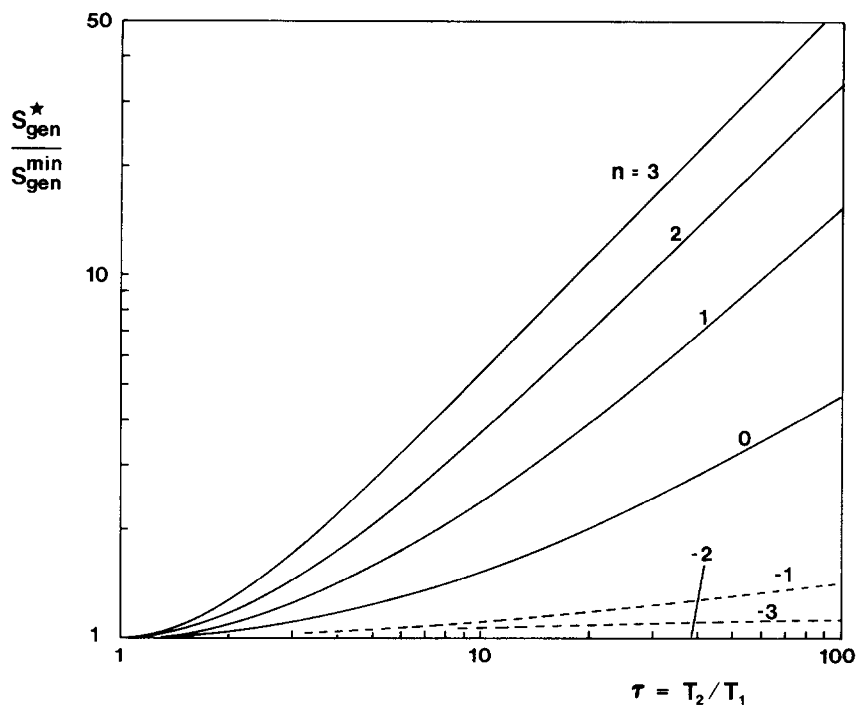

FIG 2. Relative magnitude of "constant q" system irreversibility for various values of n and τ (equation 20).

The ratio S_{gen}/S^{min}_{gen} is similar to the notion of number of entropy generation units N_S introduced by this author in the design of heat exchangers [6, 7] and energy storage units [8]. This dimensionless parameter shows the relative position of a particular design with respect to the best possible design (i.e. the least irreversible design).

In many applications, the effective conductivity $k(T)$ obeys a power law-type relationship over the temperature range covered by the insulation,

$$k \cong k_1 \tau^n. \quad (19)$$

In such cases the more general form of equation (18) becomes

$$\frac{S^*_{gen}}{S^{min}_{gen}} = \frac{n^2}{4(n+1)} \frac{(\tau-1)(\tau^{n+1}-1)}{\tau(\tau^{n/2}-1)^2} \quad (20)$$

which is also plotted on Fig. 2 for different values of n. One can easily show that the constant q design coincides with the optimum when $n = -2$; this is also evident from examining equation (8).

In conclusion, decreasing the thermal conductance Ak_1/t without providing intermediate heat transfer

Figure 3 is a schematic representation of the thermodynamic concept behind the intermediate cooling/heating effect. Large scale insulation systems are most indispensable when used in association with applications at very high temperatures (power plants) and very low temperatures (cryogenic refrigeration systems). The $r(T)$ effect is the result of placing the insulation in communication with portions of the power/refrigeration cycle. Depending on whether $r(T)$ represents cooling or heating, the insulation serves as heat source or heat sink for the thermodynamic cycle. Therefore, to be truly optimized, the insulation must become an integral part of the power or refrigeration cycle.

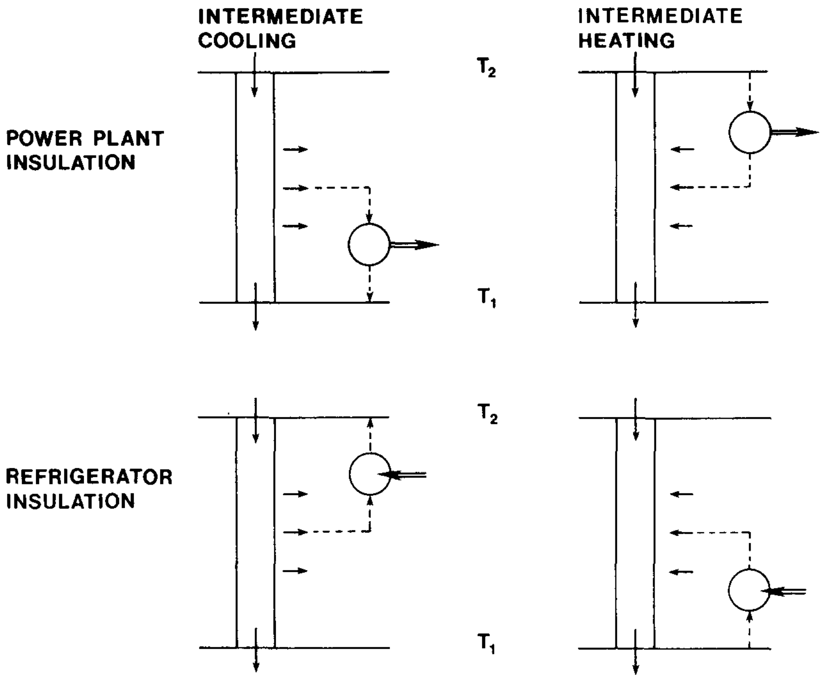

FIG. 3. Thermodynamic basis for intermediate heat exchange effect: design of insulation in harmony with the central system being insulated.

The main conclusion to be drawn from Fig. 3 is that an insulation system facing a severe temperature ratio ($\tau \gg 1$) must be optimized in harmony with the system it insulates. The distributed heat exchange effect $r(T)$ is the connection between the insulated system and its insulation.

Examples

It is appropriate to complete this section with two examples of continuous one-dimensional insulation systems whose performance is improved significantly by applying the design principle presented here. The purpose of these examples is to show that the task of thermodynamically connecting the insulation with the insulated system is possible. It is shown also that the present design philosophy is actually being used in isolated instances, although it is not being recognized as a generally valid systematic approach to optimizing any thermal insulation system.

1. As a first example, consider the class of mechanical supports which span a large temperature difference, a case considered by Bejan and Smith [9] and Bejan [10] in the context of supports for cryogenic apparata. The mechanical supports of a very hot or a very cold apparatus conduct heat between the apparatus and its environment. The heat leak can be reduced by decreasing the A/t ratio to the point where the supports are too slender to perform their mechanical function properly. Beyond this point one has to rely on the present method to further upgrade the insulation performance.

For a wide range of construction materials $k(T)$ increases with the increasing temperature so that, in equations (19), (20) and Fig. 2, n is of the order of one. The savings in useful power resulting from cooling the support at intermediate temperatures are dramatic, expecially in the case of cryogenic supports for which τ is of the order of 100. This technique is becoming popular in cryogenic engineering where it is implemented approximately, by allowing a stream of cold helium gas to flow along the support toward the warm end. In the process, the gas stream gradually intercepts a major portion of the heat leak conducted by the support. The cooling effect provided by the gas stream approximates satisfactorily the optimum required by equation (8) [9, 10].

2. As a second example, consider the counterflow heat exchanger shown in Fig. 4. A good counterflow heat exchanger is an excellent insulation system in the hot end (T_2)—cold end (T_1) direction by promoting heat transfer internally, in the stream-to-stream direction. An entropy flux analysis around the shaded element of Fig. 4 yields the local rate of entropy generation

$$\begin{aligned} dS_{gen} &= \dot{m}c_p \ln \frac{T+dT}{T} \\ &\quad + \dot{m}c_p \ln \frac{T+\Delta T}{T+\Delta T+dT} \\ &= \frac{\dot{m}c_p \Delta T}{1+\Delta T/T} \frac{dT}{T^2} \end{aligned} \tag{21}$$

where ΔT is the temperature difference between streams. Expression (21) is based on the assumption of balanced ideal gas streams and zero pressure drop in the heat exchanger passages. Comparing this result with the general formula (2) demonstrates that the group $\dot{m}c_p \Delta T/(1+\Delta T/T)$ plays exactly the same

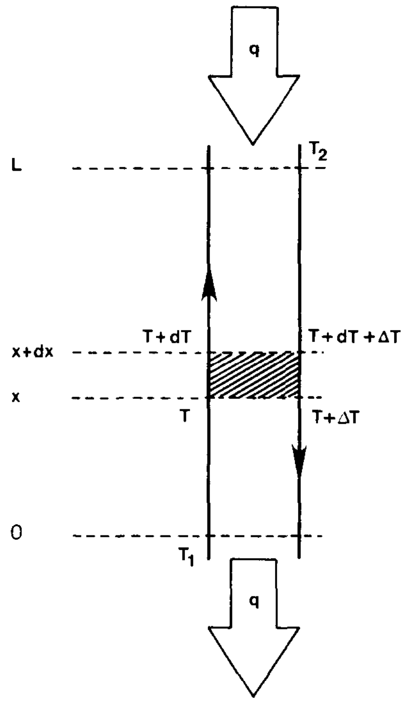

FIG. 4. Analogy between a counterflow heat exchanger and a one-dimensional thermal insulation system.

role as the heat leak q in an insulation system. Therefore, for a counterflow heat exchanger

$$q = \frac{\dot{m}c_p \Delta T}{1 + \Delta T/T}. \tag{22}$$

The effective thermal conductance for the longitudinal heat leak (22) is determined using the heat exchanger design relation [11]

$$\dot{m}c_p \mathrm{d}T = UP\Delta T \mathrm{d}x \tag{23}$$

where $\mathrm{d}x$ is the heat exchanger length element corresponding to dT, P is the perimeter wetted by one of the streams and U is the overall coefficient for stream-to-stream heat transfer based on P. Eliminating ΔT between expressions (22) and (23) and observing that, in general, $\Delta T \ll T$ yields

$$q = \frac{(\dot{m}c_p)^2}{UP} \frac{\mathrm{d}T}{\mathrm{d}x}. \tag{24}$$

Therefore, $(\dot{m}c_p)^2/(UP)$ is the equivalent of (kA) used in the general treatment of one-dimensional continuous insulations. The length of the heat exchanger, t, plays the role of insulation thickness. The longitudinal heat leak is inversely proportional to the heat-transfer area Pt, and so is the heat exchanger irreversibility.

The general conclusions reached earlier in this section apply to counterflow heat exchangers. This type of insulation is improved first by increasing the heat-transfer area Pt. When this approach is no longer feasible economically, intermediate cooling can further improve the thermodynamic performance. In many applications, the effective conductance $(\dot{m}c_p)^2/UPt$ is relatively insensitive to temperature changes, hence, the effective conductivity k is constant. Consequently, the variational principle (8) recommends a uniform intermediate cooling effect so that q becomes proportional to the absolute temperature or according to relation (22), $\Delta T/T =$ constant. This conclusion is in good agreement with the practice of cooling at intermediate temperatures the counterflow heat exchangers of helium liquefaction plants [12, 13]. The variational principle presented here provides a foundation for this procedure. In high temperature applications such as the regenerative heat exchanger for a Brayton cycle heat engine, the temperature ratio τ may be too low to make the use of intermediate cooling profitable (see, for example, the $n = 0$ curve at $\tau \cong 4$ on Fig. 2).

3. DISCONTINUOUS INSULATIONS

In this section we consider insulations whose temperature varies discontinuously from T_1 to T_2. Radiation shielding is a prime example of an insulation with discrete temperature distribution. Imagine an insulation system consisting of (N-1) radiation shields suspended in the evacuated space separating T_1 and T_2. If the number of radiation shields is very large, the insulation may be regarded as continuous and the design principle developed in the preceding section applies unchanged. If the number of shields is small so that the temperature varies appreciably between two adjacent shields, the variational principle (8) does not apply and has to be replaced with a similar principle valid for discontinuous insulations.

Consider the ith compartment separating two shields at temperatures $T(i)$ and $T(i-1)$ with a net radiant heat-transfer rate q_i. Calculating the rate of entropy generation in this compartment and summing up over all N compartments yields

$$S_{\mathrm{gen}} = \sum_{i=1}^{N} q_i \left(\frac{1}{T^{(i-1)}} - \frac{1}{T^{(i)}} \right). \tag{25}$$

The heat current q_i depends on the absolute temperatures of the two radiant surfaces,

$$q_i = \sigma A F [T^{(i)4} - T^{(i-1)4}], \tag{26}$$

where σ is Boltzmann's constant, A is the shield area and F is a factor accounting for the emissivities of the two surfaces. In general, F may also depend on $T^{(i)}$ and $T^{(i-1)}$.

The insulation irreversibility reaches its minimum when the $(N-1)$ shield temperatures are externally controlled such that

$$\frac{\partial S_{\mathrm{gen}}}{\partial T^{(i)}} = 0, \quad i = 1, 2, \ldots, N-1. \tag{27}$$

In combination with equation (25) and (26), system (27) is sufficient for determining the optimum set of shield temperatures $T^{(i)}$ or the optimum distribution of heat leak q_i which yields the minimum rate of entropy generation $S_{\mathrm{gen}}^{\mathrm{min}}$, N and A remaining fixed. This statement replaces the variational result (8) developed in Section 1 for continuous insulations with t and A fixed.

For a one-shield system, $N = 2$, the optimization program prescribed by equations (25)–(27) is relatively simple. If, for example, factor F is temperature independent, combining equations (25)–(27) leads to the following equation for the optimum shield temperature

$$4\tau_1^3\left(1+\frac{1}{\tau}\right)-6\tau_1^2-\frac{1+\tau^4}{\tau_1^2} = 0. \quad (28)$$

Here, τ_1 is the ratio $T_{opt}^{(1)}/T_1$ while $\tau = T_2/T_1$ as in the preceding section. Equation (28) can be solved numerically for τ_1 as soon as τ is specified: if $\tau = 4$, equation (28) yields $\tau_1 = 2.505$.

In systems employing two or more parallel shields the use of design criterion (27) is increasingly laborious. A more expedient way of treating such cases is by using an approximate version of the variational principle (8). As the number of shields increases, the heat-transfer rate between two adjacent shields can be approximated by

$$q_i = \sigma A F\left[T^{(i)^4} - T^{(i-1)^4}\right]$$
$$\approx 4\sigma A F T^{(i)^3}\frac{T^{(i)}-T^{(i-1)}}{i-(i-1)}$$
$$\approx 4\sigma A F T^3\frac{\mathrm{d}T}{\mathrm{d}i}. \quad (29)$$

Assuming that N is sufficiently large, we can rearrange and integrate (29) over the N gaps formed by the $N-1$ shields,

$$\frac{N}{A} = \int_{T_1}^{T_2}\frac{4\sigma F T^3}{q}\mathrm{d}T. \quad (30)$$

Comparing constraints (30) and (5) we discover that the group $4\sigma F T^3$ plays the same role as the effective conductivity k employed in developing the general method for continuous insulations. Interestingly enough, the number of spaces created between shields plays the role of insulation thickness. Substituting $(4\sigma F T^3/N)$ for (k/t) in the general optimum expressed by equation (8) we determine the optimum distribution of heat leak across the radiation shields. Combining the q_{opt} result with equation (29) yields the optimum shield temperatures

$$\tau_i = \left[1+(\tau^{3/2}-1)\frac{i}{N}\right]^{2/3}, \quad i = 1,2,\ldots,N-1 \quad (31)$$

where τ_i is shorthand for $T_{opt}^{(i)}/T_1$. In the limit $N \to \infty$ the optimum shield temperatures dictated by expressions (31) replace the temperatures obtained by solving system (27). Even in the extreme case of a one-shield insulation, for $\tau = 4$ result (31) yields $\tau_1 = 2.726$ which is reasonably close to the 2.505 estimate based on solving the corresponding equation from system (27).

We have shown that insulation systems relying on radiation shielding exhibit an effective thermal conductivity factor k which varies, approximately, as T^3. Examining the $n = 3$ curve of Fig. 2, we conclude that insulation systems of this type are prime candidates for intermediate cooling (controlled temperature distribution) as a method of effectively upgrading their thermodynamic performance.

4. INSULATIONS WITH INTERNAL HEAT GENERATION

There are insulation systems in which the heat current $q(T)$ is caused not only by the leakage of heat from the high temperature side T_2 but also by heat generated internally. The scope of this section is to develop an equivalent form of design principle (8), this time for one-dimensional continuous insulations with internal heat generation.

For example, consider the thermodynamic optimization of an electric cable connecting two regions at different temperatures (Fig. 5). The cable is also a good thermal conductor which carries a sizeable heat leak. As pointed out in the beginning of this article, the direct approach to minimizing the heat leak and the irreversibility is by decreasing the thermal conductance, making the cable long and thin. This operation cannot be carried out indefinitely: there comes a point where the electrical resistance and the overall electrical performance of the cable become unacceptable.

As in Sections 2 and 3, we are faced with the task of improving the thermal performance of an insulation with fixed geometry t and A. We accomplish this task by externally controlling the cable temperature distribution in order to minimize the rate of entropy production in the cable.

An energy balance and entropy flux analysis for the shaded $\mathrm{d}T$ element of Fig. 5 yields

$$\mathrm{d}q + \mathrm{d}w = \mathrm{d}r \quad (32)$$

$$\mathrm{d}S_{gen} = \frac{q\mathrm{d}T}{T^2}+\frac{\mathrm{d}w}{T}. \quad (33)$$

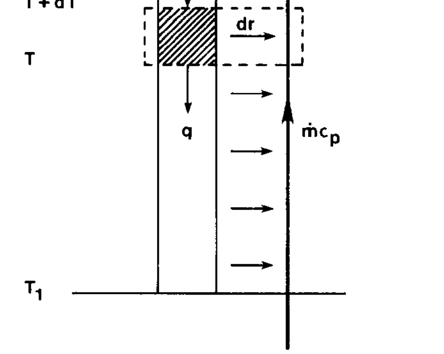

FIG. 5. Optimum intermediate cooling effect for electrical cable connecting spaces at different temperatures.

Here, dw is the electrical power dissipated locally. In an electric cable,

$$dw = \frac{\rho I^2}{A} dx, \tag{34}$$

where ρ and I are the electrical resistivity and the total electric current through a cross-section A, respectively. We should point out that in writing expression (33) we have neglected a term which accounts for the cross-coupling between the heat current q and the electric current I via the Onsager relations of irreversible thermodynamics [1, 14, 15]. The thermoelectric effect caused by this coupling is usually neglected in electric cables for power applications (high voltages and currents), as in the design example concluding this section. Expression (33) is the standard form for the rate of entropy production in a conducting medium with internal heat generation [16].

Proceeding on a path identical to the one followed in Section 2, we have to select the optimum heat leak function $q_{opt}(T)$ which minimizes the integral

$$S_{gen} = \int_{T_1}^{T_2} \left(\frac{q}{T^2} + \frac{1}{T} \frac{dw}{dT} \right) dT \tag{35}$$

subject to the geometry constraint (5). This is equivalent to minimizing the aggregate integral

$$\Phi = \int_{T_1}^{T_2} \left(\frac{q}{T^2} + \frac{1}{T} \frac{dw}{dT} + \lambda \frac{k}{q} \right) dT \tag{36}$$

subject to no constraints. The optimum distribution of heat leak is the result of solving the Euler equation

$$\frac{d}{dT} \left[\frac{\partial}{\partial q'} \left(\frac{1}{T} \frac{dw}{dT} \right) \right] - \frac{\partial}{\partial q} \left(\frac{1}{T} \frac{dw}{dT} \right) - \frac{1}{T^2} + \frac{\lambda k}{q^2} = 0 \tag{37}$$

with $q' = dq/dT$. The solution is determined as soon as the system of Fig. 5 is known and the dissipation function $w(T, q, q')$ is specified. The variational principle derived earlier, equation (7), is the special case $w = 0$ of the more general result (37).

Example

Consider a cryogenic cable transmitting a large current between a room temperature current supply ($T_2 \simeq 300\,K$) and a large scale superconducting system bathed in liquid helium ($T_1 \simeq 4\,K$). The cable physical properties may be modeled as $k = k_1$ and $\rho = bT$, where k_1 and b are both constant. This property model is reasonably good for high purity copper used in cryogenic cables (see, for example, [17]).

The group $(dw/dT)/T$ appearing in the variational principle (37) is found by combining the property model with equations (34) and (1),

$$\frac{1}{T} \frac{dw}{dT} = bI^2 \frac{k}{q}. \tag{38}$$

Solving equation (37) in combination with equation (38) we get

$$q_{opt}(T) = T(k\Lambda)^{1/2} = \left(\frac{Ak_1}{L} \ln \tau \right) T, \tag{39}$$

where $\Lambda = \lambda + bI^2$.

The optimum temperature distribution is the same as in equation (12), an exponential to be preserved for any current level I. Using expressions (32), (39) and (10) we find the intermediate cooling effect $(dr/dT)_{opt}$ required by the optimum. We point out that the required intermediate cooling can be provided very simply by forcing a coaxial gaseous-stream to flow from T_1 to T_2. In order to achieve the optimum temperature distribution (12) the stream must have the right mass flowrate. The flowrate is found by substituting dg_{opt} given by equation (39), dw given by (34) and (12) and $dr = \dot{m}c_p dT$ into the energy balance (32). The optimum flow rate is

$$\frac{\dot{m}c_p t}{Ak_1 \ln \tau} = 1 + \left[\frac{It}{A \ln \tau} \left(\frac{b}{k_1} \right)^{1/2} \right]^2, \tag{40}$$

indicating precisely how much coolant is needed as the operating current I increases. Equation (40) constitutes the optimum "operating curve" for a cryogenic cable of fixed geometry expected to carry a current which may vary from time to time.

Helium gas-cooled electrical cables have become a permanent feature of all large scale superconducting magnet systems [18]. However, the design of such cables has so far lacked the theoretical basis for systematic optimization as thermal insulation systems.

5. CONCLUDING REMARKS

Employing the concept of thermodynamic irreversibility we were able to show that the true optimization of a thermal insulation system does not end with minimizing the heat leak by decreasing the thermal conductance Ak/t. At least as important to an efficient operation is control of the heat leak distribution $q(T)$ by cooling (or heating) the insulation at intermediate temperatures.

The general treatment of three different classes of insulation systems stressed the potential savings resulting from optimum control of heat leak distribution. The absolute temperature ratio $\tau = T_2/T_1$ governs this potential. As shown in Fig. 2, intermediate heat exchange is highly recommended in insulation systems exposed to large τ's. Moderate τ applications in which the effective conductance increases sharply with the temperature will also register sizeable savings when designed with an optimally controlled distribution of heat leak. The use of second law concepts insures the general applicability of this design philosophy.

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UN PRINCIPE VARIATIONNEL GENERAL POUR LES SYSTEMES D'ISOLATION THERMIQUE

Résumé—On propose un nouveau concept pour l'optimisation des systèmes d'isolation thermique. La méthode basée sur la réduction de l'irréversibilité thermodynamique, consiste en un contrôle externe de la variation de la fuite de chaleur à travers l'isolant. On montre que les économies utiles de puissance obtenues en appliquant cette philosophie sont importantes. Cette méthode concerne en premier les systèmes d'isolation avec des rapports élevés de température absolue et ceux pour lesquels la conductivité thermique croît avec la température absolue. Trois classes de systèmes d'isolation thermique sont optimisées à partir de cette procédure universelle: isolants continus monodimensionnel, isolants avec une distribution discontinue de température et isolants continus avec génération interne de chaleur.

EIN ALLGEMEINES VARIATIONSPRINZIP FÜR DIE AUSLEGUNG VON WÄRMEISOLIERUNGEN

Zusammenfassung—In dieser Arbeit wird ein neues Konzept zur Systemoptimierung von Wärmeisolierung vorgeschlagen. Die Methode basiert auf der Minimierung der thermodynamischen Irreversibilitäten und besteht darin, die Variation der Wärmeverluste durch die Isolierung in Abhängigkeit von der Temperatur extern zu kontrollieren. Es wird demonstriert, daß bei Anwendung dieser Auslegungs-Philosophie bedeutende Energieeinsparungen festgestellt werden können. Bevorzugte Anwendungsfälle für diese Methode sind Isolierungen mit großen absoluten Temperatur-Verhältnissen und solche, in welchen die Wärmeleitfähigkeit mit der absoluten Temperatur zunimmt. Drei Klassen von Wärmeisolierungen werden nach dieser universellen Auslegungsmethode optimiert: eindimensionale homogene Isolierungen, Isolierungen mit unstetiger Temperatur-Verteilung und homogenen Isolierungen mit inneren Wärmequellen.

ОБЩИЙ ВАРИАЦИОННЫЙ ПРИНЦИП КОНСТРУИРОВАНИЯ СИСТЕМЫ ТЕПЛОВОЙ ИЗОЛЯЦИИ

Аннотация — В статье предлагается новый метод оптимизации системы тепловой изоляции, основанный на минимизации термодинамической необратимости. Метод заключается в осуществлении внешнего контроля за изменением теплопотерь с изменением температуры поперек слоя изоляции. Показано, что с помощью данного метода была получена существенная экономия энергии. Предлагаемый метод можно применять в случае изоляционных систем, испытывающих действие высоких отношений абсолютных температур, а также систем, в которых эффективная теплопроводность увеличивается с ростом значения абсолютной температуры. На основании предлагаемого универсального подхода выделено три класса теплоизоляционных систем: одномерная непрерывная изоляция, изоляция с разрывом непрерывности в распределении температур и непрерывная изоляция с внутренней генерацией тепла.